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A FREQUENCY-PARAMETRIC ANALYSIS OF NATURAL OSCILLATIONS OF NON-UNIFORM RODS†

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A constructive numerical-analytical method of investigating free transverse oscillations of a highly non-uniform rod with the boundary conditions of elastic clamping is developed. Standard special cases of the boundary conditions are also considered. To solve the corresponding self adjoint boundary eigenvalue and eigenfunction problem, an effective computational procedure for determining the frequencies and shapes of the oscillations is set up, similar to the shooting method. Assertions equivalent to the Sturm comparison theorems and corollaries of them for second-order boundary-value problems are formulated. The algorithm is tested on model examples with known solutions. A parametric synthesis is carried out for a family of conical rods for different boundary conditions, which is important for applications. The results obtained are compared with the classical results of Kirchhoff, Timoshenko and Gould.

1. FORMULATION OF THE PROBLEM

The transverse oscillations of a non-uniform rectilinear rod for different boundary conditions are investigated using an effective numerical-analytical method, which is a generalization of the well-known "shooting" method. To be specific, we will first consider the case when both ends of the rod are rigidly clamped (fastened). The corresponding self-adjoint boundary eigenvalue and eigenfunction problem (a Sturm–Liouville type problem) in dimensionless variables is described by the relations [1–4]

$$\begin{aligned} &(p(x)u'')'' = (\lambda r(x) - q(x))u, \quad 0 \le x \le l \\ &0 < p^- \le p \le p^+ < \infty, \quad 0 < r^- \le r \le r^+ < \infty, \quad 0 \le q^- \le q \le q^+ < \infty \end{aligned} \tag{1.1}$$

$$u(0) = u'(0) = u(l) = u'(l) = 0 \tag{1.2}$$

The unknown parameter $\lambda > 0$ and the function $u(x)$ characterize the frequencies and shapes of free oscillations of the rod, respectively. The coefficient $p(x)$ defines the bending stiffness, $r(x)$ is the mass per unit length, and $q(x)$ is the elasticity of the external medium. Unlike the standard approaches, the length of the rod $l > 0$ is not fixed (in particular, $l = 1$), and is assumed to be a variable parameter when constructing the solution of problem (1.1), (1.2). With respect to the properties of smoothness of the functions p, r, q we make the following extremely easy assumptions: we can confine ourselves to continuity and the possibility of extension to a certain interval $(l, \xi]$, see below.

It is required to obtain a sequence of eigenvalues $\lambda = \lambda_n$ (frequencies $\omega_n = \sqrt{\lambda_n}$) and functions $u = u_n(x)$ (the shapes of the oscillations). The main interest, from the theoretical and applied points of view, is in the lower modes of oscillations $n = 1, 2, \ldots$ (conventionally $n \leqslant 10$). It is well known that the self-adjoint boundary-value problem (1.1), (1.2) has a denumerable set of eigenvalues (a discrete spectrum) and corresponding eigenfunctions (shapes), which form an orthonormalized basis with weight $r(x)$ [2, 3]

$$\begin{aligned} &\lambda \in \{\lambda_n\}, \quad 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots, \quad \lambda_n \sim n^4 \\ &u_n(x) = u(x, \lambda_n) \quad (u_n, u_m)_r = \|u_n\|^2 \delta_{nm}; \quad n, m = 1, 2, \ldots \end{aligned} \tag{1.3}$$

Here δ_{nm} is the Kronecker delta, $(\cdot, \cdot)_r$ is the scalar product with weight $r(x)$ and $\|\cdot\|_r$ is the norm with weight. The operation of calculating the norm can be replaced by a finite operation using the "sensitivity function" $\upsilon = \partial u/\partial\lambda$ as follows [4]:

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$$\|u_n\|_r^2 = (u_n, u_n)_r = \int_0^l u_n^2(x)r(x)dx = [(pu_n'')'v_n - (pu_n'')'v_n']_{x=l}$$
$$(p(x)v'')'' = (\lambda_n r(x) - q(x))v + r(x)u_n(x), \quad v(0) = v'(0) = v''(0) = v'''(0) = 0 \tag{1.4}$$

The function $v = v_n(x)$ is the solution of the linear Cauchy problem (1.4); it is constructed numerically with the simultaneous integration of Eq. (1.1) (see below).

Problem (1.1), (1.2) can be simply reformulated in terms of the variational isoperimetric problem [2, 3]

$$J[u] = \frac{1}{2}\int_0^l (p(x)u''^2 + q(x)u^2)dx \to \min, \quad I[u] = \|u\|_r^2 = \int_0^l r(x)u^2dx = 1$$
$$u(0) = u'(0) = u(l) = u'(l) = 0 \tag{1.5}$$
$$\lambda \in \{\lambda_n\}, \quad u_n(x) = u(x, \lambda_n) \quad (u_n, u_m)_r = \delta_{nm}, \quad n = 1, 2, \ldots$$

Here λ is the doubled Lagrange multiplier and λ_1 is the value corresponding to the (global) minimum of the quadratic functional J (1.5). The subsequent quantities $\lambda_2, \lambda_3, \ldots$ correspond to the local minima of J, taking into account the orthogonality condition $(u, u_k)_r = 0, k = 1, 2, \ldots, n-1$. On the basis of a variational treatment, functional approaches have been developed for finding estimates of the quantities λ_n and the corresponding functions $u_n(x)$ [2, 3]. The computational algorithms enable us to obtain effective upper bounds λ_n^+. The construction of highly accurate lower bounds λ_n^- involves considerable computational difficulties. The existing Weinstein–Aronszajn and Fichera methods [2] are extremely cumbersome and are not very productive from the algorithmic point of view (see Section 5).

Note that rough bilateral estimates of the eigenvalues of problem (1.1), (1.2) have the form

$$\lambda_n^- \le \lambda_n \le \lambda_n^+, \quad \lambda_n^\pm = \left(\frac{\gamma_n}{l}\right)^4 \frac{p^\pm}{r^\mp} + \frac{q^\pm}{r^\mp}, \quad n = 1, 2, \ldots$$
$$\gamma_n = \operatorname{Arg}(\cos\gamma \operatorname{ch}\gamma - 1), \quad \gamma_1 = 4.7300, \quad \gamma_2 = 7.8532, \quad \gamma_3 = 10.9956 \tag{1.6}$$
$$\gamma_4 = 14.1372, \quad \gamma_5 = 17.2788, \quad \ldots, \quad \gamma_n = (n + 1/2)\pi + O(e^{-\pi n}), \quad n \gg 1$$

The bounds $\lambda_n^\pm$ (1.6) will be sufficiently accurate for λ_n (1.3), if the functions $p(x)$, $r(x)$ and $q(x)$ vary only slightly for all $x \in [0, l]$, i. e. the differences $p^+ - p^-, r^+ - r^-, q^+ - q^-$ are relatively small. In this case the perturbation method [5] can be used, which enables analytical refinement of the solution to be carried out. However, it is inefficient for numerical calculations. The bounds (1.6) can be used in the general case considered to construct an initial approximation in recurrence calculation schemes (successive approximations, accelerated convergence, etc.). These algorithms can be fairly effective in combination with procedures for extension with respect to the system parameters (parametric synthesis) or with respect to artificially introduced parameters to improve the convergence [4]. The methods and algorithms presented in the scientific literature are insufficient for carrying out highly accurate and mass operative calculations.

Note that Eq. (1.1) is inconvenient for a numerical-analytical investigation. If the function $p(x)$ is twice continuously differentiable, this equation can be reduced to a standard form of fourth-order linear equation or a first-order system of equations (the Cauchy form). However, it is more natural to introduce additional variables z and μ, which have a mechanical meaning. Problem (1.1), (1.2) takes the form

$$u' = \theta, \quad \theta' = -\frac{z}{p(x)}, \quad z' = \mu, \quad \mu' = -(\lambda r(x) - q(x))u$$
$$u(0) = \theta(0) = u(l) = \theta(l) = 0 \quad \left(u'' = -\frac{z}{p(x)}, \quad z'' = -(\lambda r(x) - q(x))u\right) \tag{1.7}$$

where z is the moment of the forces of elasticity and μ is the shearing force. The form of (1.7) does not require the coefficient $p(x)$ to be differentiable; it is sufficient for it to be continuous.

In addition to the boundary conditions of rigid clamping of the ends (1.2) or (1.7) we can take other conditions, which lead to a self adjoint boundary-value problem. These include pinning: $u = z = 0$, free ends: $z = \mu = 0$, and fixing of the direction of the tangent: $\theta = \mu = 0$. These types of conditions may exist at one or both ends. We will take as conditions of general form elastic clamping with respect to displacement and rotation of the tangent at the ends of the rod

$$[(1 - \kappa_x)\mu \mp \kappa_x u]_{x=0,l} = 0, \quad [(1 - \sigma_x)z \mp \sigma_x \theta]_{x=0,l} = 0, \quad 0 \le \kappa_{0,l}, \quad \sigma_{0,l} \le 1 \tag{1.8}$$

The normalized coefficients $\kappa_{0,l}$, $\sigma_{0,l}$ define the relative effect of the stiffness of the elastic clamping of the ends with respect to displacement and rotation, respectively. The case (1.2), (1.7) of absolutely rigid clamping corresponds to "infinitely large" values of the stiffness, i.e. the relative values are equal to unity: $\kappa_{0,l} \uparrow 1$, $\sigma_{0,l} \uparrow 1$. The special cases of boundary conditions mentioned above are limiting for (1.8) with respect to $\kappa_{0,l}$ or/and $\sigma_{0,l}$. The case of infinite stiffness with respect to displacement or/and rotation of the end can be realised rather approximately, since the material of the base, to which the ends are fastened, possesses a limited (often low) stiffness.

Note that in the equation of the oscillations of a rod (1.1) or systems (1.7), the effect of the external elastic medium (a Winkler base) is taken into account. The generalization of the model to the case of bending-torsional oscillations [6] of highly non-uniform rods is of considerable interest from the theoretical and applied points of view.

2. SCHEME OF THE SOLUTION

The standard procedure for obtaining the eigenvalues and eigenfunctions of problem (1.7) consists of constructing a general solution of the equations, which depends on the parameter λ, with subsequent satisfaction of the conditions on the boundary $x = 0, l$. The necessary and sufficient condition for the solution to be non-trivial is that the determinant of the matrix of the fundamental system should be equal to zero. This condition leads to the characteristic equation for the eigenvalues λ_n. According to the general theory, the determinant is the entire function of λ, which, in the real region where $\lambda > 0$, has a denumerable set of roots $\{\lambda_n\}$, where $\lambda_n \sim n^4$ $(n = 1, 2, \ldots)$; the bilateral estimates are represented by formulae (1.6).

According to problem (1.7), the solution is defined by the vector function $U = (u, \theta, z, \mu)$. As it applies to problem (1.7) it is required to construct two families of solutions (λ is the parameter of the family) of the Cauchy problem for the following data for $x = 0$:

$$\begin{aligned} &1)\ u(0) = \theta(0) = \mu(0) = 0, \quad z(0) = 1 \\ &2)\ u(0) = \theta(0) = z(0) = 0, \quad \mu(0) = 1 \\ &U = c_1 U_1(x, \lambda) + c_2 U_2(x, \lambda) \end{aligned} \tag{2.1}$$

The desired solution $U(x, \lambda)$ is determined, apart from a scalar factor, which is usually found from the normalization condition (1.4). Hence, in expressions (2.1) we can assume this factor to be the constant $c_1 \neq 0$ (or $c_2 \neq 0$), while the ratio $c_1/c_2 = \eta$ (or $c_1/c_2 = \eta$) together with λ can be taken to be required quantities. To calculate λ and η, we have the boundary conditions for $x = 1$

$$u_1(l, \lambda) + \eta u_2(l, \lambda) = 0, \quad \theta_1(l, \lambda) + \eta \theta_2(l, \lambda) = 0 \tag{2.2}$$

For a numerical solution of system (2.2), the sets of values $\lambda \in \Lambda$, $\eta \in H$ for each λ_n, η_n can be roughly estimated using relation (1.6). Standard numerical methods for a more accurate determination of the desired quantities require a search among a large number of versions, measure by an order of 10^{2N}, where 10^{-N} is the required relative error. Methods based on minimization of the discrepancy lead to computational difficulties ("ravine" effects), which increase as n increases, and also when there is an appreciable change in the parameters of system (1.7).

We will briefly consider a more general case of boundary conditions (1.8), corresponding to elastic clamping of the ends of the rod. The standard approach dictates the construction of four families of solutions U_i (the fundamental matrix)

$$\begin{aligned} &1)\ u(0) = 1, \quad 2)\ \theta(0) = 1, \quad 3)\ z(0) = 1, \quad 4)\ \mu(0) = 1 \\ &U(x, \lambda) = \Sigma c_i U_i(x, \lambda) \end{aligned} \tag{2.3}$$

The values of the remaining components of the solution, in addition to the ones indicated, are assumed to be equal to zero (similar to relations (2.1)). Summation over i in (2.3) is carried out from $i = 1$ to $i = 4$. The use of boundary conditions (1.8) leads to a system of equations in the required quantities λ and c_i

$$\begin{aligned} &(1-\kappa_0)c_4 - \kappa_0 c_1 = 0 \quad (1-\sigma_0)c_3 - \sigma_0 c_2 = 0 \\ &\Sigma c_i[(1-\kappa_l)\mu_i + \kappa_l u_i]_{x=l} = 0, \quad \Sigma c_i[(1-\sigma_l)z_i + \sigma_l\theta_i]_{x=l} = 0 \end{aligned} \tag{2.4}$$

As above, normalization on the non-zero constant c_j of system (2.4) leads to four equations in λ, η_i ($\eta_j = 1$). In the general situation $0 < \kappa_{0,l}, \sigma_{0,l} < 1$; then from the first two equations, the constants, for example c_3 and c_4, are expressed in terms of c_2 and c_1 respectively, and are substituted into the last two equations. After the normalization by c_1 (or c_2), these equations are reduced to the form (2.2). Various special cases of the clamping of the ends are obtained by passing to the limits as $\kappa_{0,l}$, $\sigma_{0,l} \to 1$ or $\kappa_{0,l}, \sigma_{0,l} \to 0$.

3. THE SAGITTARY FUNCTION METHOD. STURM TYPE THEOREMS

The characteristic equation for finding the eigenvalues of the parameter λ for problem (1.7) has the form

$$\begin{aligned} &S(\lambda, l) = 0, \quad \lambda = \lambda_n(l), \quad S(\lambda, x) \equiv u_1(x,\lambda)\theta_2(x,\lambda) - u_2(x,\lambda)\theta_1(x,\lambda) \\ &0 \le x \le l, \quad 0 < l < \infty, \quad \lambda > 0; \quad S(\lambda, x) > 0, \quad 0 < x \ll l, \quad \lambda \sim 1 \end{aligned} \tag{3.1}$$

As already has been noted in Section 2, the conditions for the solution of the form (2.1) of the boundary-value problem to be non-trivial, namely, that the determinant of the linear system in c_1 and c_2 should be equal to zero when $x = 1$, leads to relation (3.1). It is obvious that this relation can be obtained also by eliminating the unknown η from system (2.2). In the standard approaches the parameter l ($l = 1$) is fixed and the function S is assumed to depend only on the unknown argument λ. Below we describe methods and computational algorithms based on the idea of the sagittary function $S(\lambda, x)$, which we have introduced, that depend on two arguments, λ and x. Using it we can investigate the fundamental properties of the solution of the eigenvalue and eigenfunction problem, similar to the Sturm comparison theorems and corollaries from them [7, 8]. The meaning of the adjective sagittary and its content follow from the further constructions.

The characteristic equation and the sagittary function for boundary conditions (1.8) are defined in the same way as (3.1)

$$\begin{aligned} &S(\lambda, l) = 0, \quad \lambda = \lambda_n(l) \\ &S(\lambda, x) \equiv \kappa_0(\sigma_0 d_{34} + (1-\sigma_0)d_{24}) - (1-\kappa_0)(\sigma_0 d_{13} + (1-\sigma_0)d_{12}) \\ &d_{ij} = d_{ij}(\lambda, x) = M_i Z_j - M_j Z_i \\ &M_i = (1-\kappa_l)\mu_i + \kappa_l u_i, \quad Z_i = (1-\sigma_l)z_i + \sigma_l\theta_i, \quad i, j = 1, \ldots, 4 \end{aligned} \tag{3.2}$$

The rather lengthy representation (3.2) for the sagittary function $S(\lambda, x)$ can be reduced considerably by constructing two families of linearly independent solutions, which satisfy the following conditions when $x = 0$

$$\begin{aligned} &1)\ u(0) = 1-\kappa_0, \quad \theta(0) = 0, \quad z(0) = 0, \quad \mu(0) = \kappa_0, \quad 0 < \kappa_0 < 1 \\ &2)\ u(0) = 0, \quad \theta(0) = 1-\sigma_0, \quad z(0) = \sigma_0, \quad \mu(0) = 0, \quad 0 < \sigma_0 < 1 \\ &u = c_1u_1 + c_2u_2, \quad \theta = c_1\theta_1 + c_2\theta_2, \quad z = c_1z_1 + c_2z_2, \quad \mu = c_1\mu_1 + c_2\mu_2 \end{aligned} \tag{3.3}$$

Solution (3.3) automatically satisfies boundary condition (1.8) for $x = 0$ for arbitrary λ, c_1 and c_2. The characteristic equation and the sagittary function have the form

$$S(\lambda, l) = 0, \quad \lambda = \lambda_n(l)$$

$$S(\lambda, x) \equiv ((1-\kappa_l)\mu_1(x,\lambda) + \kappa_l u_1(x,\lambda))((1-\sigma_l)z_2(x,\lambda) + \sigma_l\theta_2(x,\lambda)) - \tag{3.4}$$

$$-((1-\kappa_l)\mu_2(x,\lambda) + \kappa_l u_2(x,\lambda))((1-\sigma_l)z_1(x,\lambda) + \sigma_l\theta_1(x,\lambda))$$

Any form of solution, from considerations of convenience, can be used in the calculations. Thus, in the first case (2.3), the construction of the fundamental system of solutions is independent of the parameters κ_0, σ_0, which are taken into account in subsequent stages of determining the sagittary function and the characteristic equation according to relations (3.2). In the second case (3.3), when constructing linearly independent solutions, the quantities κ_0, σ_0 are taken into account, but the number of integrable Cauchy problems to be integrated is halved.

Thus, in what follows, the sagittary function $S(\lambda, x)$ is further assumed to be known; it can be constructed analytically (extremely rarely) or numerically by integrating Cauchy problems. In this case either successive calculation of the functions u_i, θ_i, z_i, μ_i for fixed values of λ is required with subsequent storage, or simultaneous integration of (two or four) Cauchy problems and calculation of $S(\lambda, x)$ using finite formulae. The sagittary function S can be determined as the solution of this set of Cauchy problems and a non-linear equation. For conditions (1.7) the sagittary function S (3.1) is changed in accordance with the equation

$$S' = (z_1u_2 - z_2u_1)/p(x), \quad S(\lambda, 0) \equiv 0, \quad 0 \le x \le l$$
$$S(\lambda, x) = O(x^4), \quad S > 0, \quad 0 < x \ll l, \quad \lambda \sim 1 \tag{3.5}$$

Relations for the sagittary function, similar to (3.5), where obtained for the other boundary conditions indicated in Section 1. On the basis of an analysis of the sagittary function $S(\lambda, x)$ fundamental properties of the solution of the initial problem can be established and assertions can be formulated similar to Sturm's oscillation theorem and Sturm's comparison theorems and corollaries of them, which were developed previously for a second-order equation of the form $(pu')' + \lambda ru = 0$ with corresponding boundary conditions [2, 3, 7, 8]. These properties are very useful for approximate calculations (see Sections 4 and 5).

For a fixed value of $l > 0$ the sagittary function $S(\lambda, l)$ is an oscillating function of λ for $\lambda > 0$. If we take sufficiently large $\lambda = \lambda(l) > 0$, then $S(\lambda, x)$ is an oscillating function of x for $0 \le x \le l$, having as large a number of zeros as desired. When $r/p \ge c > 0$ for $x > 0$ the sagittary function $S(\lambda, x)$ will be an oscillating function of x independently of $\lambda > 0$. The oscillatory property also occurs as $r/p \to 0$ for $x \to \infty$, if $r/p \sim x^{-\gamma}$, $\gamma \le 4$. It should be noted that both solutions of the system $(u_1(x, \lambda), \theta_1(x, \lambda))$, $(u_2(x, \lambda), \theta_2(x, \lambda))$ (2.1) are not oscillating in this sense.

Suppose, for a fixed values of $l > 0$, we obtain a $\lambda > 0$ such that $S(\lambda, l) = 0$, where the sagittary function $S(\lambda, x)$ has no intermediate zeros with respect to x. Then the corresponding $\lambda = \lambda_1(l)$ will be the first eigenvalue of problem (1.7). If there are $n - 1 \ge 1$ intermediate zeros, then $\lambda = \lambda_n(l)$ will be the nth eigenvalue, where $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$. By analysing the properties of $S(\lambda, x)$ one can establish assertions that are useful for the approximate numerical solution of the problem.

Theorem 1. Suppose for $\lambda = \lambda^*$, the sagittary function $S(\lambda^*, x)$ has n ($n = 1, 2, \ldots$) intermediate zeros with respect x, $0 < x \le l$. Then, when $\lambda = \lambda^{**} > \lambda^*$ the sagittary function $S(\lambda^{**}, x)$ has no less than n zeros in this interval.

Moreover, an unlimited increase in λ leads to an unlimited increase in the number of zeros in the specified interval $0 \le x \le l$.

Theorem 2. Suppose for $\lambda = \lambda^*$, the sagittary function satisfies the relation $S(\lambda^*, x) = 0$ for $x = x_1$ and $x = x_2$, where $0 \le x_1 < x_2 \le l$. Then when $\lambda = \lambda^{**} > \lambda^*$ a point $x = x_3, x_1 < x_3 < x_2$ exists such that $S(\lambda^{**}, x_3) = 0$.

To construct a solution of the eigenvalue and eigenfunction problem (1.7) the following assertion regarding the distribution of the zeros of the sagittary function $S(\lambda, x)$ when $\lambda = \lambda^*, \lambda^{**}$ is constructive.

Theorem 3. Let $x = x_0$ be the common zero of the sagittary functions $S(\lambda^*, x)$ and $S(\lambda^{**}, x)$. Then, when $\lambda^{**} > \lambda^*$ the next zero $x = x_1$ for $S(\lambda^*, x)$ and $x = x_2$ for $S(\lambda^{**}, x)$ satisfies the condition $x_2 < x_1$.

Hence it follows that all the zeros $x_i > 0$ of the sagittary function $S(\lambda, x)$ are shifted to the left as λ increases, and shifted to the right as λ decreases. This property establishes a local relation between the

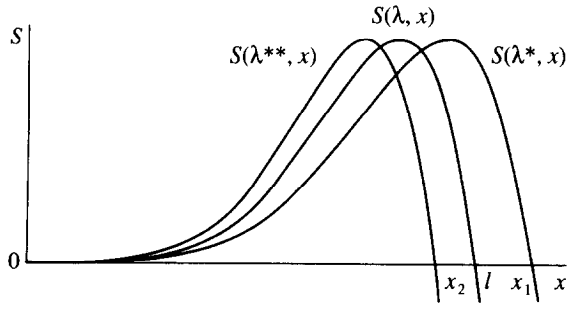

Fig. 1

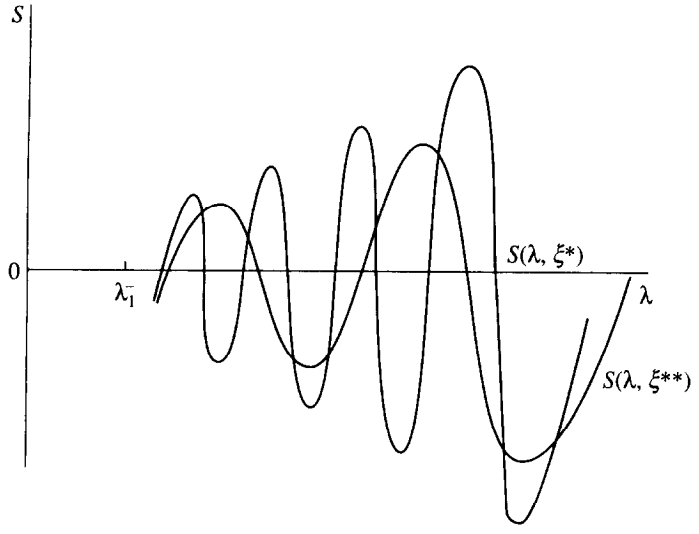

Fig. 2

required parameter λ and the length of the segment l and can be used for calculations. It enables one to determine the upper limit λ_n^+ and the lower limit λ_n^- of the eigenvalues λ_n, which is extremely important for establishing the actual accuracy of approximate solutions. In particular, for the first eigenvalue $\lambda_1(l)$ the assertion regarding the limits $\lambda_1^\pm$ and the corresponding zeros $x = \xi_1^\pm$ of the sagittary function $S(\lambda_1^\pm, x)$ holds.

Theorem 4. Let λ_1 be the upper limit of the first eigenvalue, i.e. $\lambda_1^\pm \geqslant \lambda_1(l)$. Then the sagittary function $S(\lambda_1^+, \xi^+) = 0$, where $\xi^+ \leqslant l$. Similarly, for the lower limit $\lambda_1^- \leqslant \lambda_1(l)$ we have the equality $S(\lambda_1^-, \xi^-) = 0$, where $\xi^- \geqslant l$ (when $\xi^- > l$ the functions $p(x)$, $r(x)$ and $q(x)$ are assumed to be smoothly continued in the interval $(l, \xi^-]$). If the quantities $\lambda^\pm$ are fairly close to one another, then l and $\xi^\pm$ will also be as close as desired, and moreover $S(\lambda_1^+, l) < 0$, $S(\lambda_1^-, l) > 0$.

The property $S(\lambda_1^-, x) > 0$ when $0 < x < l$ follows from simple estimates of the functions $u_{1,2}(x, \lambda_1^-)$, $\theta_{1,2}(x, \lambda_1^-)$ according to (2.1) and (3.5) (problem (1.7)). Naturally, when $n \geqslant 2$ the sagittary function $S(\lambda_n^-, x)$ may change in sign up to $(n-1)$ times, while the sagittary function $S(\lambda_n^+, x)$ changes in sign no less than n times. In the general case, the determination of the sign of the sagittary function S when $0 < x \leqslant l$ and its changes when $0 < x < l$ require lengthy estimates, which are more convenient to carry out numerically by integrating the Cauchy problems indicated above.

A typical behaviour of the sagittary function $S(\lambda, x)$ is shown qualitatively in Figs 1 and 2. The "separating" property of the sagittary function S is naturally used in computational methods, related to subsequent refinement of the required solution. These include the shooting method, the method of successive approximations, Newton-type accelerated convergence methods, etc. This is why S has been called a sagittary function (from the Latin sagitta, meaning an arrow). Note that it is preferable to use the shooting method at the initial stage of the solution of the problem using estimates of the eigenvalues

$\lambda_n^{\pm}$ obtained in advance using (1.6) or from variational estimates taking relations (1.5) into account [2–4]. These values can be refined using a Newton-type rapidly converging method if necessary. This combined approach is extremely effective in parametric synthesis, for example, when optimizing the form (see Section 6).

4. COMPUTATIONAL ALGORITHMS OF THE SHOOTING METHOD BASED ON THE SAGITTARY FUNCTION

The properties of the sagittary function $S(\lambda, x)$ (3.1) established above enable us to construct two comparatively simple shooting-type recurrence algorithms, which do not require lengthy calculations. The main consumption of processor time is due to the fact that two Cauchy problems (1.7) and (2.1) have to be integrated for a known value of the parameter λ, obtained at the previous step of the iterational procedure. Using the sagittary function $S(\lambda, x)$, the value of the parameter λ is refined and is then used at the next step of the algorithm. Then the accuracy of the solution (the discrepancy) is estimated both with respect to the "ordinate", i.e. $|S(\lambda, l)|$, and with respect to the abscissa ξ, i.e. with respect to the quantity $\delta = |l - \xi|$, where $\xi = \arg_x S(\lambda, x)$.

We will first consider the problem of finding the first eigenvalue $\lambda = \lambda_1(l)$ and, to reduce the amount of writing, we will omit the subscript. We carry out standard operations using the shooting method, similar to those used for the case of a second-order equation (the classical Sturm–Liouville problem), in which the solution $u(x, \lambda)$ has the meaning of the sagittary function.

1. Suppose we know fairly close upper and lower limits $\lambda^{\pm}$, where, by Theorem 4, the function $S(\lambda^{\pm}, l) \lesseqgtr 0$. We will devote out main attention to the closeness of λ^+ to the exact value λ, which is achieved using the Rayleigh–Ritz method. The meaning of this requirement is that there are no additional zeros of the sagittary function $S(\lambda^+, x)$, which may appear if λ^+ is fairly large (Theorem 1). The first step of the procedure involves calculating the mean value $\lambda_{(1)}$ and determining, first, the refined upper limit $\lambda^+_{(1)}$ or, second, the lower limit $\lambda^-_{(1)}$ on the basis of the separating property of the sagittary function S

$$\begin{aligned} &\lambda_{(1)} = \frac{1}{2}(\lambda^+ + \lambda^-) \\ &1)\ S(\lambda_{(1)}, l) < 0, \quad \lambda_{(1)} = \lambda^+_{(1)}, \quad \lambda^- = \lambda^-_{(1)} \\ &2)\ S(\lambda_{(1)}, l) > 0, \quad \lambda_{(1)} = \lambda^-_{(1)}, \quad \lambda^+ = \lambda^+_{(1)} \\ &\lambda^-_{(1)} \le \lambda \le \lambda^+_{(1)} \end{aligned} \tag{4.1}$$

The value of the function $S(\lambda_{(1)}, l)$ is calculated by integrating the Cauchy problems, as in Sections 2 and 3. As a result of the first step of (4.1), the interval $\delta_1 = \lambda^+ - \lambda^-$ of uncertainty of λ is halved. We then carry out the second step and determine the mean value of $\lambda_{(2)}$ and its position with respect to the exact value $\lambda_{(1)} \lesseqgtr \lambda(l)$ using the sign of the quantity $S(\lambda_{(2)}, l)$. The uncertainty in λ is thereby reduced by a factor of 4, etc. At an arbitrary ith step we have

$$\begin{aligned} &\lambda_{(i)} = \frac{1}{2}(\lambda^+_{(i-1)} + \lambda^-_{(i-1)}) \\ &1)\ S(\lambda_{(i)}, l) < 0, \quad \lambda_{(i)} = \lambda^+_{(i)}, \quad \lambda^-_{(i-1)} = \lambda^-_{(i)} \\ &2)\ S(\lambda_{(i)}, l) > 0, \quad \lambda_{(i)} = \lambda^-_{(i)}, \quad \lambda^+_{(i-1)} = \lambda^+_{(i)}; \quad \lambda^-_{(i)} \le \lambda \le \lambda^+_{(i)} \\ &\delta_i = (\lambda^+_{(i)} - \lambda^-_{(i)}) = \frac{1}{2^i}(\lambda^+ - \lambda^-), \quad i = 1, 2, \ldots \end{aligned} \tag{4.2}$$

Calculations are continued until the required relative error is obtained. One can then find the values of the discrepancies with respect to the ordinates $S(\lambda^{\pm}_{(i)}, l)$ and also with respect to the solutions $u(l, \lambda^{\pm}_{(i)})$, $\theta(l, \lambda^{\pm}_{(i)})$. One can take the quantity $\nu_i = \delta_i/(2\lambda_{(i)})$ as a measure of the relative closeness of the solution.

In addition to the elementary bisection procedure, one can use other methods, for example, the division on the section in the ratio of the "golden section". To accelerate the convergence, the interval

of uncertainty δ_{i-1} can be divided in a ratio proportional to the values of the discrepancies with respect to S, i.e.

$$\lambda_{(i)} = \lambda^-_{(i-1)} + \delta_{i-1}s^-_{i-1}/s_{i-1} = \lambda^+_{(i-1)} - \delta_{i-1}s^+_{i-1}/s_{i-1}$$
$$s^\pm_{i-1} = \left|S(\lambda^\pm_{(i-1)}, l)\right|, \quad s_{i-1} = S(\lambda^-_{(i-1)}l) - S(\lambda^+_{(i-1)}, l) \tag{4.3}$$

The procedure for refining the required solution (4.3) will have a more rapid convergence if the dependence of S on λ in the neighbourhood of the eigenvalue is close to a linear function with a non-zero slope.

The shooting method scheme described can be transferred as a whole to the case of general boundary conditions of elastic clamping of the ends (1.8). Here it is required to determine the sign-definiteness of S when $0 < x \leqslant l$. The sagittary function $S(\lambda, x)$ is constructed in accordance with the description given in Sections 2 and 3 (see (3.2)–(3.4)); it possesses the properties of separation of the zeros of ξ as a function of the parameter λ, indicated in Section 3 (see Theorems 1–4). It should be noted that the problem has a single zero root $\lambda_0 - 0$ in the case of pinning of one of the ends; the other end is free. If both ends are free, there is a double zero root $\lambda_0 = 0$. For $\lambda = \lambda_1$ the assertions presented above hold.

The next eigenvalues $\lambda_n(l)$ and eigenfunctions $u_n(x, l)$, $\theta_n(x, l)$, $z_n(x, l)$, $\mu_n(x, l)$ for $n = 2, 3, \ldots$ are constructed in the same way as described above. Here one should bear in mind the presence of $(n-1)$ intermediate zeros of the function $S(\lambda_n, x)$, $0 \leqslant x \leqslant l$ and the corresponding behaviour (sign) of the function $(\lambda^\pm_n, x)$ in the region of $x = 1$; in particular, for $n = 2$ we have $S(\lambda^\pm_2, l) \lesseqgtr 0$.

2. The recurrence algorithm of the shooting method, based on a determination of the discrepancies with respect to the abscissa – the zeros $\xi^\pm_{(i)}$ of the function $S(\lambda^\pm_{(i)}, x)$ for specified values of $\lambda = \lambda^\pm_{(i)}$ is more effective, clear, economical and stable from the computational point of view. When constructing the procedure of successive refinement of the first eigenvalue (and the next eigenvalues) the results of Theorems 3 and 4 are used. Unlike the procedure (4.1), (4.2), by finding the zeros $x = \xi^\pm_{(i)}$ we can reliably judge the existence of a solution of the boundary-value problem and its closeness to the required exact solution, corresponding to $x = \xi = l$. We can also take as a measure of the relative closeness, in addition to v_i, the quantity $\chi_i = \zeta_i/(2l)$, where $\zeta_i = \zeta^-_i - \zeta^+_i$ is the difference in the abscissa discrepancies. The scheme described does not require a determination of the signs of the function $S(\lambda^\pm_{(i)}, x)$, and involves finding the zeros $x = \xi^\pm_{(i)} > 0$ with the required number.

In particular, to calculate the first eigenvalue $\lambda = \lambda_1(l)$ we find the minimum positive roots $\xi^\pm_{(i)}$. At the first step of the recurrence procedure, similar to operations (4.1), we carry out the operations

$$\lambda_{(1)} = \frac{1}{2}(\lambda^+ + \lambda^-); \quad \arg_x S(\lambda_{(1)}, x) = \xi_1 > 0$$
$$1)\ \xi_1 < 1, \quad \lambda_{(1)} = \lambda^+_{(1)}, \quad \lambda^- = \lambda^-_{(1)}; \quad 2)\ \xi_1 > l, \quad \lambda_{(1)} = \lambda^-_{(1)}, \quad \lambda^+ = \lambda^+_1 \tag{4.4}$$
$$\lambda^-_{(1)} \leq \lambda \leq \lambda^+_{(1)}, \quad \delta_1 = \frac{1}{2}(\lambda^+ - \lambda^-) = \frac{\delta}{2}$$

According to relations (4.4), the uncertainty in the admissible values of λ is halved. In subsequent steps, it is reduced in a geometrical progression with common ratio $1/2$. An arbitrary ith step requires the following operations to be carried out (see (4.2))

$$\lambda_{(i)} = \frac{1}{2}(\lambda^+_{(i-1)} + \lambda^-_{(i-1)}); \quad \arg_x S(\lambda_{(i)}, x) = \xi_i > 0$$
$$1)\ \xi_i < 1, \quad \lambda_{(i)} = \lambda^+_{(i)}, \quad \lambda^-_{(i-1)} = \lambda^-_{(i)}; \quad 2)\ \xi_i > l, \quad \lambda_{(i)} = \lambda^-_{(i)}, \quad \lambda^-_{(i-1)} = \lambda^+_{(i)} \tag{4.5}$$
$$\lambda^-_{(i)} \leq \lambda \leq \lambda^+_{(i)}, \quad \delta_i = \frac{1}{2}\delta_{i-1} = 2^{-i}\delta, \quad i = 1, 2, \ldots, \quad \delta_0 = \delta$$

In addition to determining the relative error of the solution in terms of the parameter λ one can also find the discrepancies along the abscissa for the functions S, u, and θ. Another rule for the division of the uncertainty interval $[\lambda^-_{(i)}, \lambda^+_{(i)}]$ can be used, in particular the "golden section" ratio (see above). While in a certain small neighbourhood of the exact solution the relation $\lambda(l)$ may be linearized, by analogy with algorithm (4.3) one can divide the uncertainty interval in the ratio of the discrepancies $l - \xi^+_{i-1}$, $\xi^-_{i-1} - l$, i.e.

$$\lambda_{(i)} = \lambda^{-}_{(i-1)} + \delta_{i-1}\varepsilon^{-}_{i-1}/\varepsilon_{i-1} = \lambda^{+}_{(i-1)} - \delta_{i-1}\varepsilon^{+}_{i-1}/\varepsilon_{i-1}$$
$$\varepsilon^{\pm}_{i-1} = \left|1 - \xi^{\pm}_{i-1}/l\right|, \quad \varepsilon_{i-1} = (\xi^{-}_{i-1} - \xi^{-}_{i-1})/l \tag{4.6}$$

The refinement scheme, according to relations (4.6), can have a substantially more rapid convergence due to the substantial dependence of λ on l: $\lambda(l)$: $-\infty < \lambda'(l) \leqslant -c < 0$ $(c \sim 1)$.

The algorithm for the approximate solution (4.4)–(4.6) can be transferred as a whole to the case of the calculation of the subsequent eigenvalues λ_n and eigenfunctions $\{u_n(x, l), \theta_n(x, l), z_n(x, l), \mu_n(x, l)\}$ for $n = 2, 3, \ldots$. The abscissas $\xi^{\pm}_i$ are determined as the nth zero of the function $S(\lambda^{\pm}_i, x)$. The presence of boundary conditions of the general form (1.8) does not lead to any change in this algorithm.

The shooting-type procedures described are quite effective for preliminary moderately exact estimates of the solution. They do not lead to the accumulation of round-off errors (as in the method of successive approximations) and are stable to faults. The main consumption of processor time is due to the integration of the Cauchy problems for system (1.7). In the case of complex expressions for the coefficients $p(x)$, $r(x)$ and $q(x)$ the shooting method can turn out to be inefficient for highly accurate massive calculations over a wide range of variation of the mechanical parameters, since it requires a comparatively large number of iterations. Then, to construct the parametric synthesis it is preferable to use the somewhat more complex accelerated-convergence method [4], since it leads to a considerably smaller number of iterations. In situations where the functions $p(x)$, $r(x)$ and $q(x)$ are sharply varying functions, a considerable reduction in the integration step may be required. The choice of the most suitable algorithm for calculations can be made by additional investigation based on numerical experiments.

5. EXAMPLES

To illustrate the computational efficiency of the sagittary function method, we will consider examples of rods with strongly varying stiffness $p(x)$ and mass per unit length $r(x)$, ignoring the effect of the external medium ($q \equiv 0$). These functions are determined in a standard manner using expressions for the moment of inertia and the cross-section area of the rod respectively [1].

1. First, in order to test the algorithm we will carry out calculations for a model example [4], which allows of analytical integration of an equation of the form (1.1) for system (1.7). Suppose the ends of the rods are clamped, and the functions $p(x)$ and $r(x)$ have the for

$$p(x) = p_0(b + ax)^2, \quad r(x) = r_0(b + ax)^{-2}, \quad b + ax \neq 0 \tag{5.1}$$

Then, we have an Euler type equation [4, 7, 8], the solution of which is constructed in the form of a power function $(b + ax)^k$, where k is a complex parameter, defined as the root of the algebraic equation $k^2(k-1)^2 = \lambda/a^4$, solvable in radicals. The expression obtained for the general solution has a complicated form and contains power, trigonometric and logarithmic functions. It is difficult to obtain the roots of the characteristic equation of the form (3.1), but these can be obtained numerically for specific values of the parameters.

Note that, without loss of generality, two of the three parameters a, b and l, and also the constants p_0 and r_0 in (5.1) can be equated to unity. To be specific, we will consider the case when $ab > 0$; we can then put $a = b = 1$, $l > 0$. Solution (2.1) when $\lambda > 1/16$ takes the form

$$u = c_1u_1 + c_2u_2 = \sqrt{1+x}(c_1w_1(x, \lambda) + c_2w_2(x, \lambda)), \quad \theta = u'$$
$$w_1 = \sin(f^- h) - (f^-/f^+)\operatorname{sh}(f^+ h), \quad h = \ln(1+x), \quad 0 \leqslant x \leqslant l$$
$$w_2 = \cos(f^- h) - \operatorname{ch}(f^+ h), \quad f^{\pm} = (\sqrt{\lambda} \pm 1/4)^{1/2}, \quad \lambda > 1/16 \tag{5.2}$$
$$u \equiv 0, \quad \lambda \leqslant 1/16$$

Note that the hyperbolic functions reduce to a set of power functions of the form $(1+x)^{\pm f^{\pm}}$. It follows from relations (5.2) that the functions $u_{1,2}$, $\theta_{1,2}$ do not have an oscillating form. However, the sagittary function $S(\lambda, x)$ (3.1) is an oscillating function of both x and of λ; in fact, we have the expression

$$S(\lambda, x) = 2f^-(\cos(f^- h)\operatorname{ch}(f^+ h) - 1/4(f^+ f^-)^{-1}\sin(f^- h)\operatorname{sh}(f^+ h) - 1) \tag{5.3}$$

The characteristic equation for any $x = l > 0$ has a denumerable set of roots $\lambda_n(l)$, which can be obtained numerically taking into account the algorithms mentioned in Sections 3 and 4. In particular, for $l = 1$ we can present the "accurate" value λ_1 and the estimates λ_n

$$\lambda_1 = 2181.355, \quad \sqrt{\lambda_n} = (\pi/\ln 2)^2(n + 1/2)^2 - 1/4 + O(1/n), \quad n \gg 1 \tag{5.4}$$

We will now use the numerical algorithm described in Section 4 to determine λ_1. The upper limit, on the basis of the two-coordinate (trigonometric) approximation in the Rayleigh–Ritz method, gives the value $\lambda_1^+ = 2338.442$, which differs considerably from the exact value (5.4). Using the sagittary function method, by relations (4.4)–(4.6) we obtain the abscissa $\xi^+ = 0.9770$. Using the quantities ξ^+ we obtain the lower limit $\lambda^- = (\xi^+)^4\lambda^+ = 2131.2312$, according to the approach described previously [4]. The quantity $\lambda_{(1)}$ (4.4) turns out to be an improved upper limit, since $\xi_{(1)} < 1$. Seven iterations give the following estimates $\lambda_1^{\pm}$ of λ_1 that are extremely close to λ_1 (5.4)

$$\lambda^+_{1(7)} = 2181.4093, \quad \lambda^-_{1(7)} = 2181.3478, \quad \lambda_{1(8)} = 2181.3785$$

$$\Delta\lambda_1/\lambda_1 \sim 10^{-5}, \quad 2^{-8} = 1/256$$

We can similarly calculate approximate expressions, with the required accuracy, for the next eigenvalues λ_n, $n \geqslant 2$ and for other $l > 0$.

In the case of pinned ends of the rod ($u = z = 0$ when $x = 0.1$), using the algorithms described in Section 4, after six iterations we obtain the required estimates of the first eigenvalue

$$\lambda^+_{1(6)} = 432.3410, \quad \lambda^-_{1(6)} = 432.2820, \quad \lambda_{1(7)} = 432.3150, \quad \Delta\lambda_1/\lambda_1 \sim 10^{-4}$$

The model example described has enabled us to illustrate the main theoretical propositions and the effectiveness of the computational algorithm. Moreover, we have established that, even in cases (extremely rate) of analytical integrability (usually in terms of special functions), numerical methods have to be used. In the end it turns out to be preferable to use the numerical-analytical sagittary function method to find a solution, as described in Sections 3 and 4, without having to employ complicated analytical methods of integrating the equations.

2. With similar conditions of pinned ends of the rod, to compare the approaches we will investigate the case of Eq. (1.1) for $p(x) = 1 + 2\sin^2 \pi x$, $r(x) = 1$. This eigenvalue and eigenfunction problem was solved approximately using the Weinstein–Aronszajn method [2] for $n = 1, 2$. We obtain the values $\lambda_1 = 2.36388\,\pi^4$ and $\lambda_2 = 149.6520\,\pi^4$ with a relative error $\Delta\lambda/\lambda \sim 10^{-6}$. Using an algorithm based on sagittary functions we obtain the required estimates $\lambda_{1,2}$ after six iterations.

$$2.36387\pi^4 \leq \lambda_{1(6)} \leq 2.36389\pi^4, \quad 149.6517\pi^4 \leq \lambda_{2(6)} \leq 149.6531\pi^4$$

Note that the Weinstein–Aronszajn method is extremely lengthy and complicated for carrying out effective calculations. Our calculations confirm that the sagittary function method has considerable advantages for the class of eigenvalue and eigenfunction problems considered, described by fourth-order ordinary differential equations.

6. FREQUENCY-PARAMETRIC SYNTHESIS OF CONICAL RODS

We will investigate the natural oscillations of a rod of conical form. The geometrical and inertial characteristics of a truncated cone are given by the expressions

$$\begin{gathered} R(x) = R_0(1 - \alpha x/l), \quad G(x) = \pi R^2(x), \quad I(x) = (\pi/4)R^4(x) \\ V_\alpha = V_1(3 - 3\alpha + \alpha^2), \quad 0 \leq \alpha \leq 1, \quad p(x) = EI(x), \quad r(x) = dG(x) \end{gathered} \tag{6.1}$$

Here R is the radius, G is the area, I is the moment of inertia of the transverse cross-section of the rod, α is a coefficient (the tangent of the semi-aperture angle), V_α is the volume of the rod, V_1 is the volume when $\alpha = 1$ ($V_1 = 1/3\pi R_0^2 l$), E is Young's modulus of the material and d is the (volume) density. The values of all the parameters in (6.1), apart from α, are assumed to be fixed. By introducing

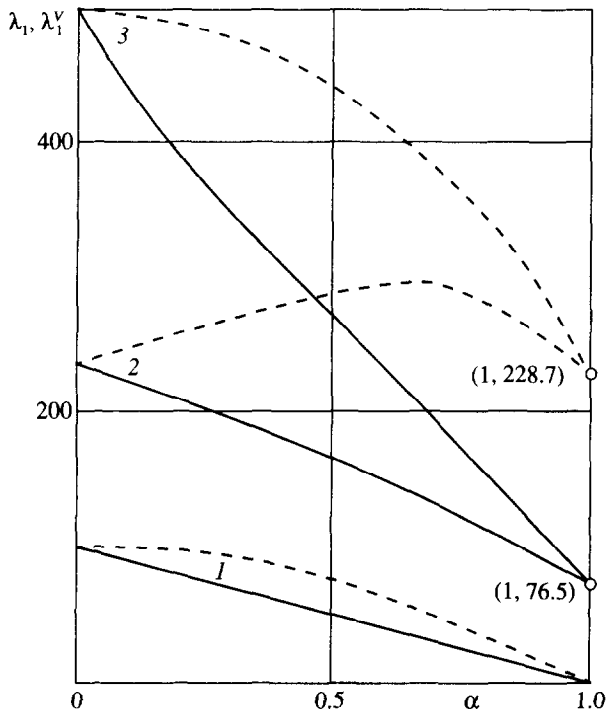

Fig. 3

dimensionless parameters and the argument x, $0 \leqslant x \leqslant 1$, Eqs (1.1) and (6.1) can be reduced to a form in which the stiffness and the density per unit length are

$$p(x) = (1 - \alpha x)^4, \quad r(x) = (1 - \alpha x)^2$$

In what follows, we will consider the natural oscillations of a family of conical rods of fixed length and radius for $x = 0$ and varying volume. The case $\alpha = 0$ corresponds to a uniform rod of cylindrical form, $\alpha = 1$ corresponds to a tapered conical rod, for which, when $x = l\ (= 1)$ there is a singularity ($p(1) = r(1) = 0$) in Eq. (1.1) and system (1.7). Note that when $\alpha \uparrow 1$ it necessarily follows that $\xi_i \alpha < 1$.

An investigation of the natural oscillations of conical rods for different boundary conditions is of interest for applied problems. We will consider three versions of the boundary conditions:

(1) both ends of the rod are pinned

$$u(0) = u''(0) = U(1) = u''(1) = 0 \tag{6.2}$$

(2) the left end of the rod is rigidly clamped while the right end is simply supported (pinned)

$$u(0) = u'(0) = u(1) = u''(1) = 0 \tag{6.3}$$

(3) both ends are rigidly clamped

$$u(0) = u'(0) = u(1) = u'(1) = 0 \tag{6.4}$$

The results of calculations of the first eigenvalue $\lambda_1(\alpha)$, corresponding to the fundamental mode of oscillations, are shown in Fig. 3. The continuous curves 1, 2, 3 correspond to conditions (6.2), (6.3), (6.4). The calculations were carried out for $0 \leqslant \alpha \leqslant 0.95$ using the sagittary-function method with a relative error of $\Delta\lambda/\lambda_1 \sim 10^{-4}$. Note that when $\alpha = 0.95$, the coefficients $p(x)$ and $r(x)$ change considerably (by a factor of 1.6×10^5 and 400) as x changes from $x = 0$ to $x = 1$. Highly accurate calculations can be carried out using the accelerated-convergence method [4] for $\alpha = 0.999$ and larger values ($\alpha \to 1 - 0$).

The results of calculations and curve 1 confirm that as $\alpha \uparrow 1$ the eigenvalue (the fundamental oscillation frequency) of a pinned rod approaches zero. This corresponds to free rotation of the rod around the

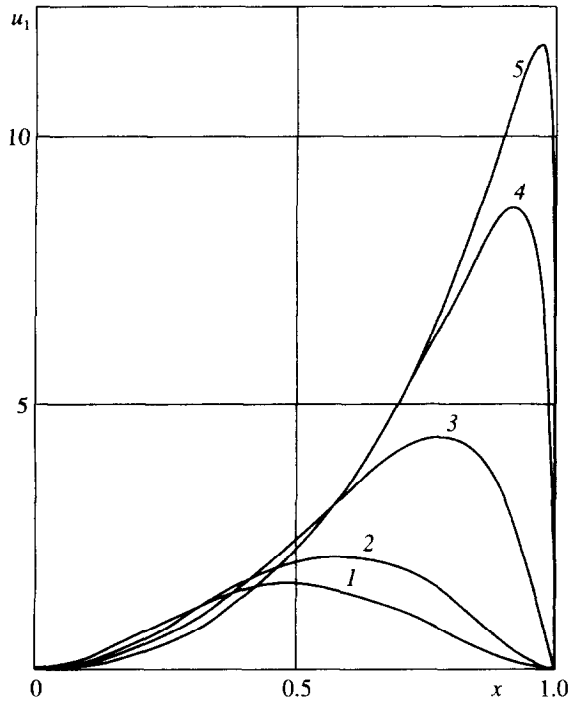

Fig. 4

hinge axis at $x = 0$; there is no elastic reaction at the end $x = 1$, since the radius tends to zero. The curve of $\lambda_1(\alpha)$ is close to a straight line, where $\lambda_1(0) = \pi^4$, $\lambda_1(1) = 0$.

In case 2, when the left end ($x = 0$) is clamped, while the right end ($x = 1$) is pinned, the first eigenvalue (frequency) is considerably greater (graph 2 is higher than graph 1). As $\alpha \uparrow 1$ the value of $\lambda_1(\alpha)$ tends from above to the value $\lambda_1(1) \approx 76.5$, which corresponds to a cantilever fixed beam of conical form [1, 4]. Curve 2 also has a comparatively simple form, but is more clearly seen to have a convexity upwards.

In the case of rigidly clamped ends (curve 3) the eigenvalues $\lambda_1(\alpha)$ are considerably greater, but as $\alpha \uparrow 1$ they rapidly converge to the value mentioned above of $\lambda_1(1) \approx 76.5$, corresponding to a cantilever clamped conical rod. The graph of $\lambda_1(\alpha)$ is extremely close to a straight line. The eigenfunctions $u_1(x, \lambda)$, normalized with weight $r(x)$, are shown in Fig. 4 for $\alpha_i = 0, 0.5, 0.9, 0.99$ and 0.999.

Similarly, using the algorithm described in Section 4, and using the idea and properties of the sagittary function, presented in Section 3, we calculated the next eigenvalues $\lambda_n(\alpha)$ (the oscillation frequencies) of a conical rod, and also carried out calculations for other types of boundary conditions.

We will consider the problem of constructing graphs of $\lambda(\alpha)$ in another case, which is of interest when designing the shape of rods.

Suppose the length of the rod l and its volume $V_\alpha = V_0$ are fixed. Then the radius R_0 of the cross-section of the rod at $x = 0$ will be a function of the parameter α

$$R(x) = R_0(\alpha)(1 - \alpha x/l), \quad R_0(\alpha) = (V_0/\pi l)^{1/2}(1 - \alpha + \alpha^2/3)^{-1/2} \tag{6.5}$$

Hence, at $x = 0$ the radius $R(x)$ of the cross-section of the conical rod $R(0) = R_0(\alpha)$, so that the volume $V_\alpha \equiv V_0 = \text{const}$; as α increases from $\alpha = 0$ to $\alpha = 1$ the radius increases by a factor of 3. By substituting expressions (6.5) into (6.1) we can transform the coefficients of Eq. (1.1) or (1.7) so that the eigenvalues $\lambda(\alpha)$ and $\lambda^V(\alpha)$ of the initial problem ($R_0 = \text{const}$) and the transformed problem ($V_\alpha = V_0$) are connected by the simple relation

$$\lambda^V(\alpha) = (1 - \alpha + \alpha^2/3)^{-1}\lambda(\alpha), \quad 0 \le \alpha < 1 \tag{6.6}$$

The functions $\lambda^V(\alpha)$ are given for the fundamental oscillation mode, in the cases considered of boundary conditions (6.2), (6.3), (6.4), in Fig. 3 (the dashed curves 1, 2, 3). For $\alpha = 0$ the values of λ^V and λ are identical, which is the case according to (6.6); for $0 < \alpha < 1$ the curves of $\lambda^V(\alpha)$ are considerably

higher than the curves of $\lambda(\alpha)$. This fact is also fairly obvious: a similar cone of large volume for these boundary conditions has a higher oscillation frequency. It is interesting to note that in the case of asymmetric (different) boundary conditions (6.3) a pronounced maximum of the function $\lambda^V(\alpha)$ is observed when $\alpha \approx 0.7$.

The above examples confirm that applied shape-optimization problems can be effectively investigated using the numerical–analytical methods developed, both in parametric and complete variational formulations.

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